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# TOPOLOGY VIA LOGIC

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## CONTENTS

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Preface	
Notation	
1 Introduction	1
<i>A Historical Overview</i>	
2 Affirmative and refutative assertions	5
<i>In which we see a Logic of Finite Observations and take this as the notion we want to study.</i>	
3 Frames	12
<i>In which we set up an algebraic theory for the Logic of Finite Observations: its algebras are frames.</i>	
3.1 Algebraicizing logic	12
3.2 Posets	13
3.3 Meets and joins	14
3.4 Lattices	18
3.5 Frames	21
3.6 Topological spaces	22
3.7 Some examples from computer science	23
<i>Finite observations on bit streams</i>	23
<i>Different physical assumptions</i>	27
<i>Flat domains</i>	28
<i>Function spaces</i>	29
3.8 Bases and subbases	31
3.9 The real line	32
3.10 Complete Heyting algebras	34

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4	Frames as algebras	38
	<i>In which we see methods that exploit our algebraicizing of logic.</i>	
4.1	Semilattices	38
4.2	Generators and relations	39
4.3	The universal characterization of presentations	42
4.4	Generators and relations for frames	46
5	Topology: the definitions	52
	<i>In which we introduce Topological Systems, subsuming topological spaces and locales.</i>	
5.1	Topological systems	52
5.2	Continuous maps	54
5.3	Topological spaces	57
	<i>Spatialization</i>	59
5.4	Locales	60
	<i>Localification</i>	62
5.5	Spatial locales or sober spaces	64
5.6	Summary	67
6	New topologies for old	70
	<i>In which we see some ways of constructing topological systems, and some ways of specifying what they construct.</i>	
6.1	Subsystems	70
6.2	Sublocales	71
6.3	Topological sums	76
6.4	Topological products	80
7	Point logic	89
	<i>In which we seek a logic of points, and find an ordering and a weak disjunction.</i>	
7.1	The specialization preorder	89
7.2	Directed disjunctions of points	92
7.3	The Scott topology	95
8	Compactness	98
	<i>In which we define conjunctions of points and discover the notion of compactness.</i>	
8.1	Scott open filters	98
8.2	The Hofmann–Mislove Theorem	100
8.3	Compactness and the reals	103
8.4	Examples with bit-streams	105
8.5	Compactness and products	106

## Contents

---

8.6	Local compactness and function spaces	110
9	Spectral algebraic locales	116
	<i>In which we see a category of locales within which we can do the topology of domain theory.</i>	
9.1	Algebraic posets	116
9.2	Spectral locales	119
9.3	Spectral algebraic locales	121
9.4	Finiteness, second countability and $\omega$ -algebraicity	125
9.5	Stone spaces	127
10	Domain Theory	134
	<i>In which we see how certain parts of domain theory can be done topologically.</i>	
10.1	Why domain theory?	134
10.2	Bottoms and lifting	136
10.3	Products	138
10.4	Sums	139
10.5	Function spaces and Scott domains	142
10.6	Strongly algebraic locales (SFP)	146
10.7	Domain equations	152
11	Power domains	165
	<i>In which we investigate domains of subsets of a given domain.</i>	
11.1	Non-determinism and sets	165
11.2	The Smyth power domain	166
11.3	Closed sets and the Hoare power domain	169
11.4	The Plotkin power domain	171
11.5	Sets implemented as lists	176
12	Spectra of rings	181
	<i>In which we see some old examples of spectral locales.</i>	
12.1	The Pierce spectrum	181
12.2	Quantales and the Zariski spectrum	182
12.3	Cohn's field spectrum	185
	Bibliography	191
	Index	196

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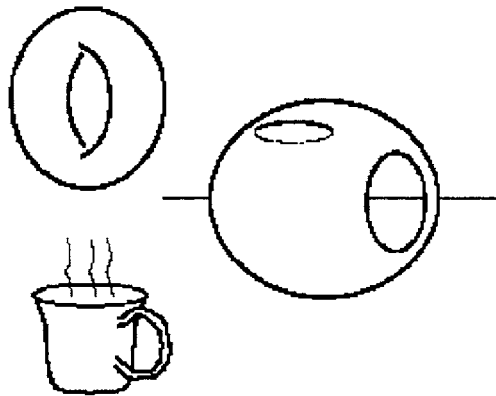
## INTRODUCTION

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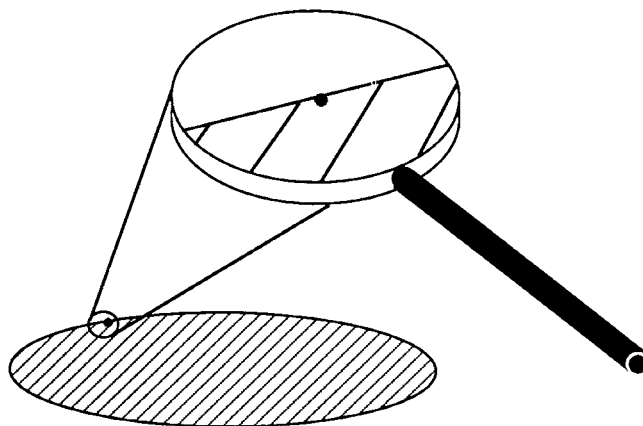
### *A Historical Overview*

The origins of topology are very different from the context we shall be working in, and it is probably as well to compare some different ideas of what it is.

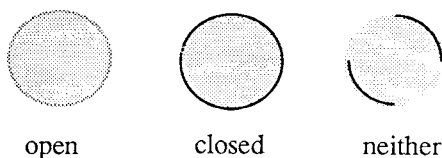
I – The first idea is that of *rubber sheet geometry*, that is to say geometry in which we don't mind stretching our space. This geometry is not at all concerned with distances or angles; it wants to answer questions like, "Is there a hole in this object?" (Although stretching is allowed, tearing isn't.) Martin Gardner [66] says, "Topologists have been called mathematicians who do not know the difference between a cup of coffee and a doughnut," the reason being that each has exactly one hole through it. According to rubber sheet geometry they are equivalent, because if they are made of stretchy enough material, one can be manipulated into the other. (The hole in the cup that counts is the one where you put your finger to hold it. The place where the coffee goes is a mere hollow. Of course, we are thinking of *ring* doughnuts.)



II – *The study of boundaries*. Since tearing is what makes a difference in rubber sheet topology, and since tearing creates new *boundaries* in the sheet, these would seem an important thing to look at. The characteristic of a boundary point of a set is that however closely you look at it, you can see some neighbouring points inside the set and some outside.



Part of this will entail studying *closed* sets, which include all their boundary points (like a circle with its circumference), and *open* sets, which include none of their boundary points (like a circle without its circumference). Of course, these are just extreme cases. There will also be sets that include some of their boundary points but not all.



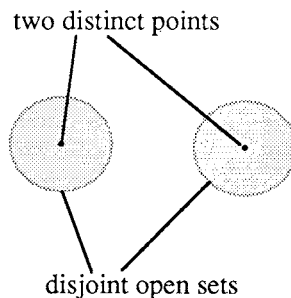
III – *The abstract study of open and closed sets.* The next step is one of abstraction. We forget all the geometry and just take an abstract set of “points”, a *topological space*. We specify certain subsets as being *open* (their complements are the *closed* subsets), and we make sure that certain axioms, due to Hausdorff, are satisfied. Then we translate topological arguments from stage II into this abstract setting.

IV – *Locale theory.* The next step is to forget even about the points, and just take an abstract set of “open sets”, with abstract algebraic operations to represent union and intersection. This structure is a *frame*. Sometimes, the points can be reconstructed from this frame of open sets.

This may seem like the ultimate in abstraction, but we shall see how considerations of *logic* make this an appropriate starting point from which to work backwards.

## Hausdorff spaces

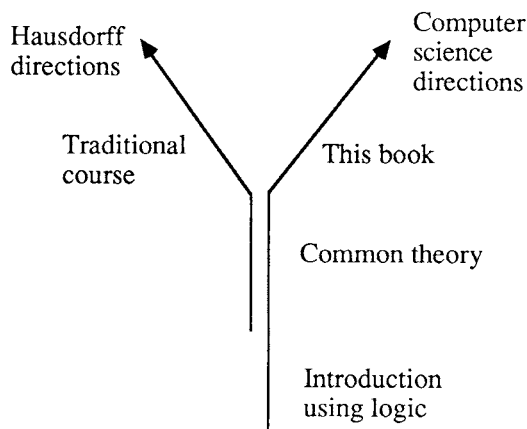
The topological axioms for open sets are very general and cover far more situations than just those arising from the rubber sheet ideas. Therefore in practice, topologists will apply extra axioms to restrict attention to the kind of space they're interested in. A very common one is the *Hausdorff separation axiom*, which says that any two distinct points can be “housed orf” from each other by disjoint open sets. The mainstream of topology deals with these *Hausdorff spaces*.



In computer science, however, topology is used to explain *approximate* states of information: the points include both approximate points and more refined points, and these relate to the topology by the property that if an open set contains an approximate point then it must also contain any refinement of it. Thus the approximate point and its refinement cannot possibly be “housed orf” by disjoint open sets and the topological space cannot possibly be Hausdorff. This means that topology as used in computer science – at least for the methods described here – runs in a different direction from the mainstream, even though it is still topology.

This book approaches topology in an unusual way, starting from frames and an explanation in terms of logic, and ends up with unusual applications – the non-Hausdorff topologies used in computer science. For computer scientists it is designed to provide a self-contained introduction, but as a route to the more traditional applications it is written to complement the standard introductions, of which there are many.





## Other books to read

For rubber sheet geometry – browse through Martin Gardner’s collections from his Scientific American column “Mathematical Puzzles and Diversions”. Gardner [63] and [86] both contain relevant articles.

For a traditional approach to topology, giving greater emphasis to Hausdorff spaces, there is a wide choice of texts. A standard one is Kelley’s “General Topology” [55].

For more on locale theory, an excellent book is Johnstone’s “Stone Spaces” [82]. However, the later chapters do assume a good acquaintance with traditional topology.

## Category theory

Interspersed throughout the text are remarks such as “categorically speaking, ...”. These refer to category theory, and readers familiar with this will understand. The rest can ignore the remarks if they want. They indicate that we are, in a hidden way, using the methods of category theory. It is not necessary to know category theory to be able to understand this book, and in fact it is probably useful to see the methods in action informally before going on to the formal theory. The classic introduction is MacLane’s “Categories for the Working Mathematician” [71], but a helpful one for computer scientists is the tutorial part of Pitt et al. [85].

Such remarks, and also those on other topics that are slightly off the main development, are often printed in a smaller typeface.